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Effect of Bacterial Memory Dependent Growth by Using Fractional Derivatives Reaction-Diffusion Chemotactic Model

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Abstract In this paper, numerical solutions of a reaction-diffusion chemotactic model of fractional orders for bacterial growth will be present. A new solution is constructed in power series. The fractional derivatives are described in the Caputo sense. We compare the experimental result obtained with those obtained by simulation of the chemotactic model without fractional derivatives. The results show that the solution continuously depends on the time-fractional derivative. The resulting solutions spread faster than the classical solutions and may exhibit asymmetry, depending on the fractional derivative used. We present results of numerical simulations to illustrate the method, and investigate properties of numerical solutions. The Adomian's decomposition method (ADM) is used to find the approximate solution of fractional 'reaction-diffusion chemotactic model. Numerical results show that the approach is easy to implement and accurate when applied to partial differential equations of fractional order.

Keywords Adomian's decomposition method · Bacteria growth · Fractional calculus

1 Introduction

In recent years, fractional derivatives have received considerable interest in recent years. In many applications, they provide more accurate models of systems under consideration. For example, they have been used successfully to model frequency dependent damping behavior of many viscoelastic materials. Other authors have demonstrated applications of fractional

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derivatives in the areas of electrochemical processes [1, 2], dielectric polarization [3], colored noise [4], viscoelastic materials [5–8] and chaos [9]. Mainardi [10] and Rossikhin and Shitikova [11] presented survey of the application of fractional derivatives, in general to solid mechanics, and in particular to modeling of viscoelastic damping. Magin [12–14] presented a three part critical review of applications of fractional calculus in bioengineering. Applications of fractional derivatives in other fields and related mathematical tools and techniques could be found in [12–22]. In fact, real world processes generally or most likely are fractional order systems.

The study of pattern formation in bacterial colonies is of particular interest both from the biological and physical points of view [23]. Growth and development of bacteria population shows a great variety of geometrical shapes. Adler [24, 25] was the first obtained bacterial population waves in the form of concentric circle. Golding et al. [26] are interested in the opposite case, where nutrient supply is limited. For a more realistic description of colonial development on a nutrient-poor surface, they take into account the interaction of bacteria with the nutrient field $s(x,t)$. Chemotaxis means changes in the movement of the cell in response to a gradient of certain chemical fields [27–30]. The movement is biased along the gradient either in the gradient direction or in the opposite direction. Miyata and Sasaki [31] estimate the distance between spots generated by the bacteria colony model with chemotactic activity.

Let $b(x,t)$ and $s(x,t)$ be the densities of the active bacteria, and the concentration of substrate. The classical one-dimensional 'reaction-diffusion equation model for b and s is [32]:

$$\begin{aligned} \frac{\partial b}{\partial t} &= \nabla \cdot (D_b(b,s)\nabla b) - \nabla \cdot (b\chi(s)\nabla s) + g(s)b \\ \frac{\partial s}{\partial t} &= D_s \Delta s - g(s)b \end{aligned} \tag{1.1}$$

where $D_b(b,s)$ is the nonlinear diffusion coefficient for bacterial cells, D_s is the constant diffusion coefficient for the substrate, $g(s)$ is function describing the specific rate of bacterial growth, and $\chi(s)$ describing the chemotactic response. The known model of chemotaxis, developed by Keller and Segel [33], was modified for the case of nonlinear diffusion. The diffusion coefficient for bacterial cells was not a fixed parameter; but it is a function of the cell density and the nutrient concentration. It was proposed and studied by Kawasaki et al. [34], which took $D_b(b,s) = Dbs$. They justify this form by the observation that bacteria are active mostly at the edge of the colony—the only area where there is high amount of bacteria and food. Their model, too, exhibits branching shapes. This is due to the b dependence of the diffusion coefficient, which leads to front instability, just as in the Kitsunezaki model. The fact that D_b also depends on s prevents bacteria inside the colony from moving—and closing the dips created by the instability. In this way, branches are created without a need for a death term.

The main aim of this paper is to solve mathematical model of a 'reaction-diffusion model of fractional-orders for chemotactic model of bacteria colonies:

$$\begin{aligned} \frac{\partial^\alpha b}{\partial t^\alpha} &= \nabla \cdot (D_b(b,s)\nabla b) - \nabla \cdot (b\chi(s)\nabla s) + g(s)b \\ \frac{\partial^\alpha s}{\partial t^\alpha} &= D_s \Delta s - g(s)b \end{aligned} \tag{1.2}$$

We implemented the Adomian decomposition method (ADM) [35, 36] to this model subject to the initial conditions

$$\begin{aligned} b(x,0) &= b_0(x) \\ s(x,0) &= v_0 \end{aligned} \tag{1.3}$$

where v_0 is arbitrary constant. In this paper we present a solution of a more general model of fractional reaction-diffusion (FRD) systems (1.2). The system is obtained from the standard diffusion-reaction systems (1.1) by replacing the first time derivative term by a fractional derivative of order.

The concept of fractional or non-integer order derivation and integration can be traced back to the genesis of integer order calculus itself [37]. Almost most of the mathematical theory applicable to the study of non-integer order calculus was developed through the end of 19th century. However it is in the past hundred years that the most intriguing leaps in engineering and scientific application have been found. The calculation technique has in some cases had to change to meet the requirement of physical reality. The use of fractional differentiation for the mathematical modeling of real world physical problems has been widespread in recent years, e.g. the modeling of earthquake, the fluid dynamic traffic model with fractional derivatives, measurement of viscoelastic material properties, etc.

The reason of using fractional order differential equations (FOD) is that FOD are naturally related to systems with memory which exists in most biological systems. Also they are closely related to fractals which are abundant in biological systems. The results derived of the fractional system (1.2) are of a more general nature. Respectively, solutions to the fractional diffusion equation spread at a faster rate than the classical diffusion equation, and may exhibit asymmetry. However, the fundamental solutions of these equations still exhibit useful scaling properties that make them attractive for applications.

The derivatives are understood in the Caputo sense. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses.

The Adomian's decomposition method will be applied for computing solutions to the systems of fractional partial differential equations considered in this paper. This method has been used to obtain approximate solutions of a large class of linear or nonlinear differential equations. It is also quite straightforward to write computer codes in any symbolic languages. The method provides solutions in the form of power series with easily computed terms. It has many advantages over the classical techniques mainly; it provides efficient numerical solutions with high accuracy, minimal calculations.

Cherruault [38] proposed a new definition of the method and he then insisted that it will become possible to prove the convergence of the decomposition method. Cherruault and Adomian [39] proposed anew convergence series. A new approach of the decomposition method was obtained in a more natural way than was given in the classical presentation [40].

Recently, the application of the method is extended for fractional differential equations [41–44].

2 Fractional Calculus

There are several approaches to the generalization of the notion of differentiation to fractional orders e.g. Riemann–Liouville, Grünwald–Letnikow, Caputo and Generalized Func-

tions approach [45]. Riemann–Liouville fractional derivative is mostly used by mathematicians but this approach is not suitable for real world physical problems since it requires the definition of fractional order initial conditions, which have no physically meaningful explanation yet. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations [45]. Unlike the Riemann–Liouville approach, which derives its definition from repeated integration, the Grünwald–Letnikow formulation approaches the problem from the derivative side. This approach is mostly used in numerical algorithms.

Here, we mention the basic definitions of the Caputo fractional-order integration and differentiation, which are used in the up coming paper and play the most important role in the theory of differential and integral equation of fractional order.

The main advantages of Caputo's approach are the initial conditions for fractional differential equations with Caputo derivatives take on the same form as for integer order differential equations.

Definition 2.1 The fractional derivative of $f(x)$ in the Caputo sense is defined as:

$$D^{\alpha} f(x) = I^{m-\alpha} D^{m} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha+1} f^{(m)}(t)\, dt$$

for $m-1 < \alpha \leq m$, $m \in N$, $x > 0$.

For the Caputo derivative we have $D^{\alpha} C = 0$, C is constant

$$D^{\alpha} t^{n} = \left\{ \begin{array}{ll} 0 & (n \leq \alpha - 1) \\ \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} t^{n-\alpha} & (n > \alpha - 1) \end{array} \right\}$$

Definition 2.2 For m to be the smallest integer that exceeds α, the Caputo fractional derivatives of order $\alpha > 0$ is defined as

$$D^{\alpha} u(x,t) = \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \left\{ \begin{array}{ll} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha+1} \frac{\partial^{m} u(x,\tau)}{\partial \tau^{m}}\, d\tau & \text{for } m-1 < \alpha < m \\ \frac{\partial^{m} u(x,t)}{\partial t^{m}} & \text{for } \alpha = m \in N \end{array} \right\}$$

3 Analysis of the Method

To give the approximate solution of nonlinear fractional-order differential equations by means of the ADM, we write the systems in the form

$$\begin{aligned} D^{\alpha_1} u_1(x,y,t) &= N_1(u_1, u_2, \ldots, u_m) + f_1(x,y,t) \\ D^{\alpha_2} u_2(x,y,t) &= N_2(u_1, u_2, \ldots, u_m) + f_2(x,y,t) \\ &\vdots \\ &\vdots \\ D^{\alpha_m} u_m(x,y,t) &= N_m(u_1, u_2, \ldots, u_m) + f_m(x,y,t) \end{aligned} \tag{3.1}$$

where D^{α_i} $(i = 1, 2, \ldots, m)$ are the fractional operators, and $N_1, N_2, \ldots, N_m$ are nonlinear operators.

Applying the inverse operators $I^{\alpha_1}, I^{\alpha_2}, \ldots, I^{\alpha_M}$ to the systems (3.1)

$$\begin{aligned} u_1(x,y,t) &= I^{\alpha_1}\big(N_1(u_1,u_2,\ldots,u_m) + f_1(x,y,t)\big) \\ u_2(x,y,t) &= I^{\alpha_2}\big(N_2(u_1,u_2,\ldots,u_m) + f_2(x,y,t)\big) \\ &\vdots \\ u_m(x,y,t) &= I^{\alpha_M}\big(N_m(u_1,u_2,\ldots,u_m) + f_m(x,y,t)\big) \end{aligned} \tag{3.2}$$

Subject to the initial conditions

$$u_i(x,y,0) = g_i(x,y) \quad (i = 1,2,\ldots,m) \tag{3.3}$$

The Adomian decomposition method suggests that the linear terms $u_i(x,t)$ are decomposed by an infinite series of components

$$u_i(x,y,t) = \sum_{n=0}^{\infty} u_{i,n}(x,y,t) \quad (i = 1,2,\ldots,m) \tag{3.4}$$

and the nonlinear operators are defined by the infinite series of the so called Adomian polynomials

$$N_i = \sum_{n=0}^{\infty} A_{i,n} \tag{3.5}$$

where $u_{i,n}(x,t); n \geq 0$ are the components of $u_i(x,t)$, that will be elegantly determined, and $A_{i,n}; n \geq 0$ are Adomian's polynomials that can be generated for all forms of non-linearity [46]. Substituting (3.4) and (3.5) into (3.2) gives

$$\begin{aligned} \sum_{n=0}^{\infty} u_{1,n}(x,y,t) &= g_1(x,y) + I^{\alpha_1}\left(\sum_{n=0}^{\infty} A_{1,n} + f_1(x,y,t)\right) \\ \sum_{n=0}^{\infty} u_{2,n}(x,y,t) &= g_2(x,y) + I^{\alpha_2}\left(\sum_{n=0}^{\infty} A_{2,n} + f_2(x,y,t)\right) \\ &\vdots \\ &\vdots \\ \sum_{n=0}^{\infty} u_{m,n}(x,y,t) &= g_m(x,y) + I^{\alpha_m}\left(\sum_{m=0}^{\infty} A_{m,n} + f_m(x,y,t)\right) \end{aligned} \tag{3.6}$$

Following Adomian analysis, the nonlinear system (3.1) is transformed into a set of recursive relations given by

$$\begin{aligned} u_{i,0}(x,y,t) &= g_i(x,y), \\ u_{i,n+1}(x,y,t) &= I^{\alpha}\big(A_{i,n} + f_i(x,y,t)\big) \quad n \geq 0 \quad (i = 1,2,\ldots,m) \end{aligned} \tag{3.7}$$

It is an essential feature of the decomposition method that the zeroth components $u_{i,0}(x,y,t)$ are defined always by all terms that arise from initial data and from integrating the inhomogeneous terms. The remaining pairs $(u_{i,n}, n \geq 1)$ can be easily determined in

a parallel manner. Additional pairs for the decomposition series normally account for higher accuracy. Have been determined the components of $u_i(x, y, t)$, the solutions of the system follow immediately in the form of a power series expansion upon using (3.4). The series obtained can be summed up in many cases to give a closed form solution for concrete problems, the n term approximants can be used for numerical purposes. Comparing the scheme presented above with existing techniques such as characteristics method and Riemann invariants, it is clear that the decomposition method introduces a fundamental qualitative difference in approach, because no assumptions are made. The approach is straightforward and the rapid convergence is guaranteed. To give a clear overview of the content of this work, several illustrative examples have been selected to demonstrate the efficiency of the method.

4 Linear Stability Analysis of Deterministic System

In this section, we consider the linear stability of the deterministic system:

$$\begin{aligned} \frac{\partial^\alpha b}{\partial t^\alpha} &= \nabla \cdot \big(D_b f(b,s)\nabla b\big) - \nabla \cdot \big(b\chi(s)\nabla s\big) + g(b,s) \\ \frac{\partial^\alpha s}{\partial t^\alpha} &= D_s \nabla^2 s - g(b,s) \end{aligned} \tag{4.1}$$

where D_b, D_s are the diffusion coefficients describing the bacterial movement and nutrient, and $0 < \alpha \leq 1$, $D_b \neq D_s > 0$.

4.1 Proposition

The necessary and sufficient conditions for the presence of fractional order Turing instability of the system (4.1) are:

(i) Under the existence of diffusion, and Chemotaxis the system is unstable.
(ii) The system without diffusion and Chemotaxis is asymptotically stable.

Proof Let $b = b^* + e^{ikx}\tilde{b}$, $s = s^* + e^{ikx}\tilde{s}$, into (4.1) and neglecting terms $O(\tilde{b}^2)$ and $O(\tilde{s}^2)$, we obtain

$$\begin{aligned} \frac{\partial^\alpha \tilde{b}}{\partial t^\alpha} &= \big(g_b^* - k^2 D_b f^*\big)\tilde{b} + \big(g_s^* + k^2 b^* \chi^*\big)\tilde{s} \\ \frac{\partial^\alpha \tilde{s}}{\partial t^\alpha} &= \big(-g_b^*\big)\tilde{s} + \big(-k^2 D_s - g_s^*\big)\tilde{s} \end{aligned}$$

where $g^* = g(b^*, s^*) = 0$, $g_b^* = g_b(b^*, s^*)$, $f^* = f(b^*, s^*)$, $\chi^* = \chi(s^*)$.

We obtain the systems

$$D^\alpha \tilde{w} = \tilde{A} w$$

where

$$\tilde{w} = \begin{pmatrix} \tilde{b} \\ \tilde{s} \end{pmatrix}, \qquad A = \begin{pmatrix} g_b^* - k^2 D_b f^* & g_s^* + k^2 b^* \chi^* \\ -g_b^* & -k^2 D_s - g_s^* \end{pmatrix}$$

To get the eigenvalues form the equation

$$|A - \lambda I| = 0$$

then, we obtain

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \mathrm{Det}(A) = 0 \tag{4.2}$$

where

$$\mathrm{Tr}(A) = g_b^* - g_s^* - k^2\big(D_b f^* + D_s\big)$$
$$\mathrm{Det}(A) = k^4 D_b D_s f^* + k^2\big(D_b f^* g_s^* + b^* \chi^* - D_s g_b^*\big)$$

and each $(\cdot)^*$ is calculated at the steady states b^* and s^*.

From the standard Turing Instability theory, in order to form the nonuniform spatiotemporal pattern, the following conditions are required:

(i) Under the existence of diffusion and chemotaxis, the system is unstable.
(ii) The system without diffusion and chemotaxis is asymptotically stable.

To prove (i) According to Descarte's rule of signs it follows that (4.2) has two positive roots or it has two imaginary roots with positive real parts.

Under this rule, we solve (4.2) we find that it has two imaginary roots with positive real parts, and then the roots are

$$\lambda = \frac{\mathrm{Tr}(A) \pm i\sqrt{4\,\mathrm{Det}(A) - (\mathrm{Tr}(A))^2}}{2}$$

If we use the results $|\arg(\lambda)| < \frac{\alpha\pi}{2}$ then, the system is unstable when

$$\left|\tan^{-1}\frac{\sqrt{4\,\mathrm{Det}(A) - (\mathrm{Tr}(A))^2}}{\mathrm{Tr}(A)}\right| < \frac{\alpha\pi}{2}, \quad \mathrm{Tr}(A) > 0,\ 4\,\mathrm{Det}(A) > \big(\mathrm{Tr}(A)\big)^2$$

To prove (ii) Let $D_b = D_s = 0$ then $\mathrm{Det}(A) = 0$—the eigenvalues form the equation $|A - \lambda I| = 0$.

Then, we obtain $\lambda^2 - \mathrm{Tr}(A)\lambda = 0 \Rightarrow \lambda(\lambda - \mathrm{Tr}(A)) = 0$, $\lambda_0 = 0$, $\lambda_1 = \mathrm{Tr}(A) = g_b^* - g_s^*$. Then the system is stable when $\mathrm{Tr}(A) < 0$, i.e.

$$\mathrm{Tr}(A) = g_b^* - g_s^* = s^* - b^* < 0 \quad \text{where } g(b^*, s^*) = b^* s^*$$

Then the system is stable when $s^* < b^*$. □

5 Applications and Numerical Results

In order to illustrate the advantages and the accuracy of the ADM for solving nonlinear fractional chemotactic model of bacteria colonies, we consider time-fractional chemotactic

model of bacteria colonies (1.1) in two dimensions with

$$D_b(b,s) = Db_s, \qquad g(s) = k\frac{s}{1+s}, \qquad \chi(s) = vs$$

As

$$\begin{aligned}
\frac{\partial^\alpha b}{\partial t^\alpha} &= D\left(\frac{\partial}{\partial x}\left(bs\frac{\partial b}{\partial x}\right) + \frac{\partial}{\partial y}\left(bs\frac{\partial b}{\partial y}\right)\right) - v\left(\frac{\partial}{\partial x}\left(bs\frac{\partial s}{\partial x}\right) + \frac{\partial}{\partial y}\left(bs\frac{\partial s}{\partial y}\right)\right) + k\frac{bs}{1+s} \\
\frac{\partial^\alpha s}{\partial t^\alpha} &= D_s\left(\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2}\right) - k\frac{bs}{1+s}
\end{aligned} \tag{5.1}$$

The coefficient of linear diffusion for bacterial cells D and the chemotaxis coefficient v were regarded as free parameters and varied in numerical results with the model. Note that D is inversely related to the density of bacterial cells (the denser the bacterial medium, the slower the diffusion), whereas v directly controls the magnitude of their chemotactic response, k is an arbitrary constant.

Applying the inverse operators I^α to the system (5.1)

$$\begin{aligned}
b(x,y,t) = I^\alpha\Bigg(&D\left(\frac{\partial}{\partial x}\left(bs\frac{\partial b}{\partial x}\right) + \frac{\partial}{\partial y}\left(bs\frac{\partial b}{\partial y}\right)\right) \\
&- v\left(\frac{\partial}{\partial x}\left(bs\frac{\partial s}{\partial x}\right) + \frac{\partial}{\partial y}\left(bs\frac{\partial s}{\partial y}\right)\right) + k\frac{bs}{1+s}\Bigg) \\
s(x,y,t) = I^\alpha&\left(D_s\left(\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2}\right) - k\frac{bs}{1+s}\right)
\end{aligned} \tag{5.2}$$

subject to the initial conditions [30]

$$\begin{aligned}
b(x,y,0) &= a_0 \exp\left(-\frac{x^2+y^2}{a}\right) \\
s(x,y,0) &= v_0
\end{aligned} \tag{5.3}$$

where v_0, a, a_0 are constants. The Adomian's decomposition method suggests that the linear terms $b(x,y,t)$, and $s(x,y,t)$ are decomposed by an infinite series of components

$$\begin{aligned}
b(x,y,t) &= \sum_{n=0}^{\infty} b_n(x,y,t) \\
s(x,y,t) &= \sum_{n=0}^{\infty} s_n(x,y,t)
\end{aligned} \tag{5.4}$$

Substituting (5.4) into (5.2) gives

$$\begin{aligned}
\sum_{n=0}^{\infty} b_n(x,y,t) = b_0(x,y) + I^\alpha\Bigg\{&D\left(\frac{\partial}{\partial x}\sum_{n=0}^{\infty} A_n + \frac{\partial}{\partial y}\sum_{n=0}^{\infty} B_n\right) \\
&- v\left(\frac{\partial}{\partial x}\sum_{n=0}^{\infty} C_n + \frac{\partial}{\partial y}\sum_{n=0}^{\infty} D_n\right) + k\sum_{n=0}^{\infty} E_n\Bigg\}
\end{aligned} \tag{5.5}$$

$$\sum_{n=0}^{\infty} s_n(x, y, t)$$
$$= s_0(x, y) + I^{\alpha}\left\{ D_s\left(\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} s_n(x,t)\right) + \frac{\partial^2}{\partial y^2}\left(\sum_{n=0}^{\infty} s_n(x,t)\right) - k\sum_{n=0}^{\infty} E_n\right)\right\} \quad (5.6)$$

and the nonlinear operator is defined by the infinite series of the so called Adomian's polynomials

$$bb_x s = \sum_{n=0}^{\infty} A_n$$
$$bb_y s = \sum_{n=0}^{\infty} B_n$$
$$bss_x = \sum_{n=0}^{\infty} C_n$$
$$bss_y = \sum_{n=0}^{\infty} D_n$$
$$\frac{bs}{1+s} = \sum_{n=0}^{\infty} E_n$$

where $b_n(x, y, t)$, and $s_n(x, y, t)$; $n \geq 0$ are the components of $b(x, y, t)$, and $s(x, y, t)$ that will be elegantly determined.

Following Adomian analysis, the nonlinear system (5.1) is transformed into a set of recursive relations given by

$$\begin{aligned} b_0(x, y, t) &= b_0(x, y) \\ b_{n+1}(x, y, t) &= I^{\alpha}\left(D\left(\frac{\partial}{\partial x}A_n + \frac{\partial}{\partial y}B_n\right) - v\left(\frac{\partial}{\partial x}C_n + \frac{\partial}{\partial y}D_n\right) + kE_n\right), \quad n \geq 0 \end{aligned} \quad (5.7)$$

and

$$\begin{aligned} s_0(x, t) &= v_0 \\ s_{n+1}(x, t) &= I^{\alpha}\left(D_s\left(\frac{\partial^2}{\partial x^2}(s_n) + \frac{\partial^2}{\partial y^2}(s_n)\right) - kE_n\right), \quad n \geq 0 \end{aligned} \quad (5.8)$$

where A_n, B_n, C_n, D_n, E_n are the Adomian's polynomial calculated for all forms of nonlinearity according to specific algorithms constructed by Adomian as

$$\begin{aligned} A_0 &= s_0 b_0 (b_0)_x \\ A_1 &= (s_0 b_1 + s_1 b_0)(b_0)_x + s_0 b_0 (b_1)_x \\ A_2 &= (s_1 b_1 + s_0 b_2 + s_2 b_0)(b_0)_x + (s_1 b_0 + s_0 b_1)(b_1)_x + s_0 b_0 (b_2)_x \end{aligned} \quad (5.9)$$

and

$$\begin{aligned} B_0 &= s_0 b_0 (b_0)_y \\ B_1 &= (s_0 b_1 + s_1 b_0)(b_0)_y + s_0 b_0 (b_1)_y \\ B_2 &= (s_1 b_1 + s_0 b_2 + s_2 b_0)(b_0)_y + (s_1 b_0 + s_0 b_1)(b_1)_y + s_0 b_0 (b_2)_y \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} C_0 &= s_0 b_0 (s_0)_x \\ C_1 &= (s_0 b_1 + s_1 b_0)(s_0)_x + s_0 b_0 (s_1)_x \\ C_2 &= (s_1 b_1 + s_0 b_2 + s_2 b_0)(s_0)_x + (s_1 b_0 + s_0 b_1)(s_1)_x + s_0 b_0 (s_2)_x \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} D_0 &= s_0 b_0 (s_0)_y \\ D_1 &= (s_0 b_1 + s_1 b_0)(s_0)_y + s_0 b_0 (s_1)_y \\ D_2 &= (s_1 b_1 + s_0 b_2 + s_2 b_0)(s_0)_y + (s_1 b_0 + s_0 b_1)(s_1)_y + s_0 b_0 (s_2)_y \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} E_0 &= \frac{b_0 s_0}{1+s_0} \\ E_1 &= \frac{b_1 s_0}{1+s_0} + \frac{b_0 s_1}{(1+s_0)^2} \\ E_2 &= \frac{b_2 s_0}{1+s_0} + \frac{b_0 s_2 + b_1 s_1}{(1+s_0)^2} - \frac{b_0 s_1^2}{(1+s_0)^3} \end{aligned} \tag{5.13}$$

By using ((5.7)–(5.13)), we could be able to calculate some of the terms of the decomposition series (5.4) as:

$$b_0(x, y, t) = a_0 e^{-\frac{x^2+y^2}{a}}$$

$$\begin{aligned} b_1 &= \left(\frac{2a_0^2 v_0 D}{a^2}(4x^2 + 4y^2 - 2a) e^{-\frac{2}{a}(x^2+y^2)} + \frac{k a_0 v_0}{1+v_0} e^{-\frac{1}{a}(x^2+y^2)} \right) \frac{t^\alpha}{\Gamma(\alpha+1)} \\ b_2 &= \frac{a_0 v_0}{a^4 (1+v_0)^3} e^{-\frac{3}{a}(x^2+y^2)} \big(a^4 k^2 v_0 (1+v_0) e^{\frac{2}{a}(x^2+y^2)} \\ &\quad - a^2 a_0 k \big(a^2 k + 4a(3D+v) v_0 (1+v_0)^2 - 8(3D+v) v_0 (1+v_0)^2 (x^2+y^2) \big) e^{\frac{1}{a}(x^2+y^2)} \\ &\quad + 4a_0^2 D (1+v_0)^2 \big(a^3 k - 108 a D v_0 (1+v_0)(x^2+y^2) + 72 D v_0 (1+v_0)(x^2+y^2)^2 \big) \\ &\quad + a^2 \big(20 D v_0 (1+v_0) - 3k(x^2+y^2) \big) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \end{aligned}$$

and

$$\begin{aligned} s_0 &= v_0 \\ s_1 &= \left(\frac{-k a_0 v_0}{1+v_0} e^{-\frac{1}{a}(x^2+y^2)} \right) \frac{t^\alpha}{\Gamma(\alpha+1)} \end{aligned}$$

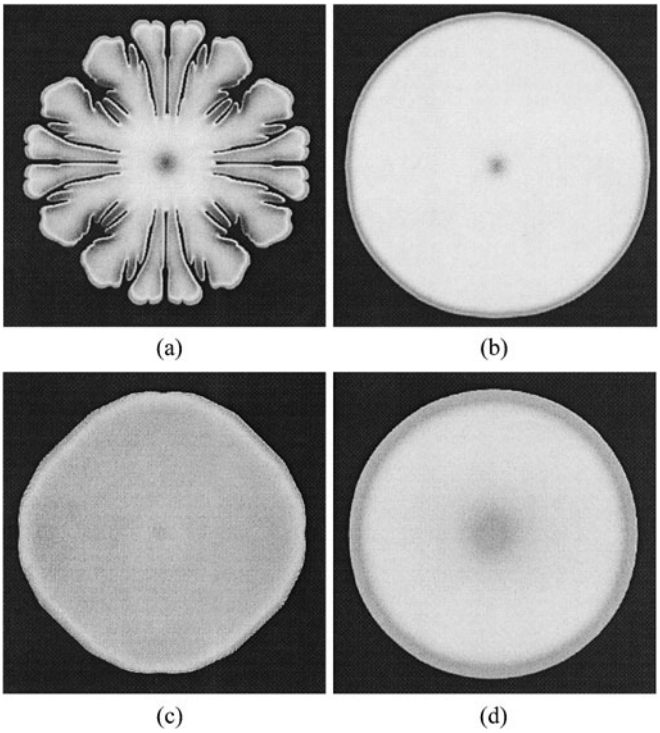

Fig. 1 Spatial density distribution of bacterial cells (Aslanidi et al. results) for various values of diffusion and chemotaxis coefficients D and v: (**a**) $D = 0.1$, $v = 2.0$; (**b**) $D = 0.1$, $v = 10.0$; (**c**) $D = 0.5$, $v = 2.0$; and (**d**) $D = 0.5$; $v = 10.0$

$$
\begin{aligned}
s_2 = -\frac{a_0 k v_0}{a^2(1+v_0)^3} e^{-\frac{2}{a}(x^2+y^2)} \big((1+v_0)\big(a^2 k v_0 - 4aD_s(1+v_0) \\
+ 4D_s(1+v_0)(x^2+y^2)\big) e^{\frac{1}{a}(x^2+y^2)} + a_0\big(8Dv_0(1+v_0)^2(x^2+y^2)^2 \\
- 4aDv_0(1+v_0)^2 - a^2 k\big)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}
\end{aligned}
$$

and so on, substituting $b_0, b_1, b_2, \ldots$, and $s_0, s_1, s_2, \ldots$ into (5.4) gives the solution $b(x, y, t)$ and $s(x, y, t)$ in a series form by:

$$
\begin{aligned}
b(x, y, t) &= b_0 + b_1 + b_2 + \cdots \\
s(x, y, t) &= s_0 + s_1 + s_2 + \cdots
\end{aligned}
\tag{5.14}
$$

See Figs. 1, 2, and Tables 1–3.

6 Discussion and Comment

We employ the Adomian's decomposition method form as a reasonable basis for studying the solution of chemotactic model of bacteria colonies. Simulation results are shown in

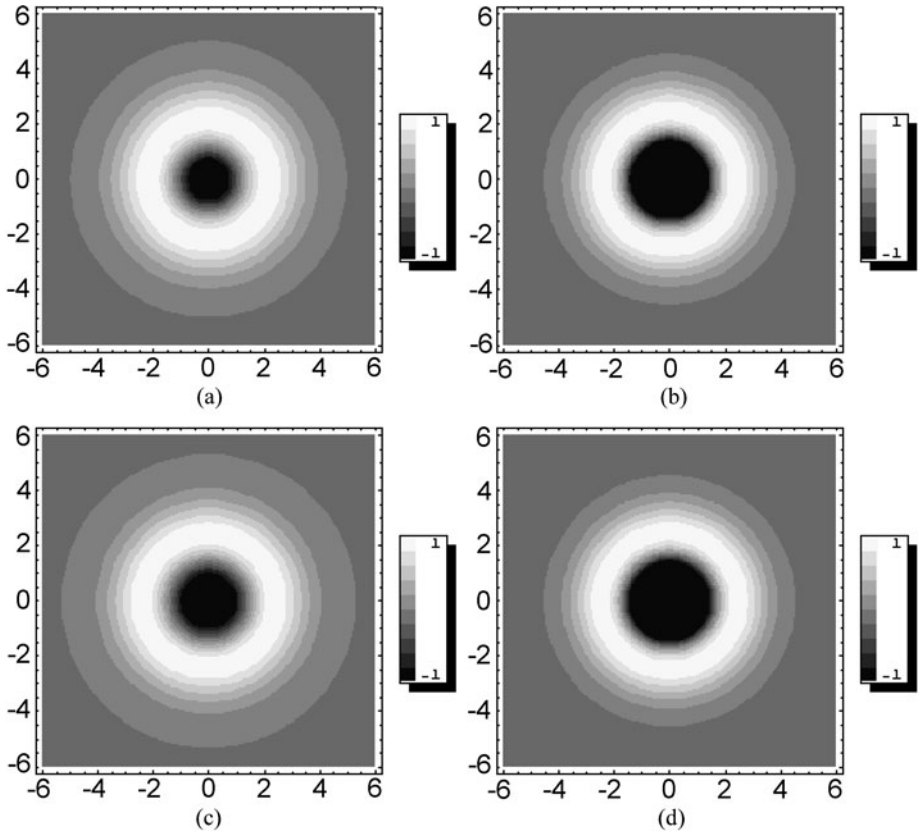

Fig. 2 The movement of bacteria cells, black color corresponds to high cell density. Parameters used in simulations at $\alpha \to 1$: (**a**) $D = 0.1$, $v = 2.0$; (**b**) $D = 0.1$, $v = 10.0$; (**c**) $D = 0.5$, $v = 2.0$; and (**d**) $D = 0.5$; $v = 10.0$, $t = 220$

Table 1 The numerical results at $\alpha = 1$, $a = 6.25$, $a_0 = 0.5$, $D_s = 1$, $k = 0.1$, $v_0 = 1.5$, $t = 0.1$

x	y				
	0	1	2	3	4
0	4.914878E–01	4.226908E–01	2.660811E–01	1.204758E–01	3.919018E–02
1	4.226908E–01	3.628342E–01	2.274326E–01	1.026634E–01	3.337429E–02
2	2.660811E–01	2.274326E–01	1.413318E–01	6.345265E–02	2.061468E–02
3	1.204758E–01	1.026634E–01	6.345265E–02	2.842199E–02	9.241864E–03
4	3.919018E–02	3.337429E–02	2.061468E–02	9.241864E–03	3.010047E–03

Fig. 2. Figure 2 are time evolution of the bacteria density $b(x, y, t)$. In Fig. 2, value of the population density $b(x, y, t)$ is shown as the gradations of black to white, for example, the region with black color denotes the high-density population one. With fixed value of chemotaxis v, we can see that a high-density bacterial spreads out to the center of the region. After that, the high-density ring appears on the first stage and concentric high density bacterial

Table 2 The numerical results at $\alpha = 0.99$

x	y				
	0	1	2	3	4
0	4.91272E–01	4.22597E–01	2.66145E–01	1.20533E–01	3.92054E–02
1	4.22597E–01	3.62822E–01	2.27508E–01	1.02712E–01	3.33867E–02
2	2.66145E–01	2.27508E–01	1.41397E–01	6.34807E–02	2.06213E–02
3	1.20533E–01	1.02712E–01	6.34807E–02	2.84320E–02	9.24422E–03
4	3.92054E–02	3.33867E–02	2.06213E–02	9.24422E–03	3.01066E–03

Table 3 The numerical results at $\alpha = 0.95$

x	y				
	0	1	2	3	4
0	4.90367E–01	4.22198E–01	2.66415E–01	1.20776E–01	3.92706E–02
1	4.22198E–01	3.62767E–01	2.27826E–01	1.02920E–01	3.34396E–02
2	2.66415E–01	2.27826E–01	1.41675E–01	6.36008E–02	2.06494E–02
3	1.20776E–01	1.02920E–01	6.36008E–02	2.84749E–02	9.25427E–03
4	3.92706E–02	3.34396E–02	2.06494E–02	9.25427E–03	3.01328E–03

colonies are formed as shown in Fig. 2(a)–(d) which is very similar to the experimental result in Fig. 1.

On the other hand with fixed value of D (Fig. 2), the process at the first stage is similar to the second case (Fig. 2(a)–(d)). We observed that, with fixed D values and increased v values; the population is more active than fixed v and increase D. In Tables 1–3, numerical results shows that when the order of the fractional derivative decreases, the density of bacteria decreases. These results shows that the bacteria growth depends not only by the chemotaxis coefficient v and coefficient of linear diffusion for bacterial D, but also continuously depends on the fractional derivative.

Finally, the decomposition method introduces a significant improvement in this field over existing techniques. The corresponding numerical solutions are obtained according to the recurrence relation using Mathematica.

References

1. Ichise, M., Nagayanagi, Y., Kojima, T.: An analog simulation of non-integer order transfer functions for analysis of electrode processes. J. Electronical. Chem. Interfacial Electrochem. **33**, 253–265 (1971)
2. Sun, H.H., Onaral, B., Tsao, Y.: Application of positive reality principle to metal electrode linear polarization phenomena. IEEE Trans. Biomed. Eng. **BME-31**(10), 664–674 (1984)
3. Sun, H.H., Abdelwahab, A.A., Onaral, B.: Linear approximation of transfer function with a pole of fractional order. IEEE Trans. Automat. Control **AC-29**(5), 441–444 (1984)
4. Mandelbrot, B.: Some noises with 1/f spectrum, a bridge between direct current and white noise. IEEE Trans. Inform. Theory **13**, 289–298 (1967)
5. Bagley, R.L., Calico, R.A.: Fractional order state equations for the control of viscoelastic structures. J. Guid. Control Dyn. **14**, 304–311 (1991)
6. Koeller, R.C.: Application of fractional calculus to the theory of viscoelasticity. J. Appl. Mech. **51**, 299–307 (1984)
7. Koeller, R.C.: Polynomial operators. Stieltjes convolution and fractional calculus in hereditary mechanics. Acta Mech. **58**, 251–264 (1986)

8. Skaar, S.B., Michel, A.N., Miller, R.K.: Stability of viscoelastic control systems. IEEE Trans. Automat. Contr. **AC-33**(4), 348–357 (1988)
9. Hartley, T.T., Lorenzo, C.F., Qammar, H.K.: Chaos in a fractional order Chua system. IEEE Trans. Circuits Syst. I **42**(8), 485–490 (1995)
10. Mainardi, F.: Fractional calculus: some basic problem in continuum and statistical mechanics. In: Carpinteri, A., Mainardi, F. (eds.) Fractals and Fractional Calculus in Continuum Mechanics, pp. 291–348. Springer, Wien/New York (1997)
11. Rossikhin, Y.A., Shitikova, M.V.: Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids. Appl. Mech. Rev. **50**, 15–67 (1997)
12. Magin, R.L.: Fractional calculus in bioengineering. Crit. Rev. Biomed. Eng. **32**, 1–104 (2004)
13. Magin, R.L.: Fractional calculus in bioengineering—part 2. Crit. Rev. Biomed. Eng. **32**, 105–193 (2004)
14. Magin, R.L.: Fractional calculus in bioengineering—part 3. Crit. Rev. Biomed. Eng. **32**, 194–377 (2004)
15. Oldham, K.B., Spanier, J.: The Fractional Calculus. Academic Press, New York (1974)
16. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives-Theory and Applications. Gordon & Breach, Longhorn (1993)
17. Podlubny, I.: Fractional Differential Equations. Academic Press, New York (1999)
18. El-Sayed, A.M.A., Rida, S.Z., Arafa, A.A.M.: On the solutions of time-fractional bacterial chemotaxis in a diffusion gradient chamber. Int. J. Nonlinear Sci. **7**, 485–492 (2009)
19. El-Sayed, A.M.A., Rida, S.Z., Arafa, A.A.M.: On the solutions of the generalized reaction-diffusion model for bacteria growth. Acta Appl. Math. **110**, 1501–1511 (2010)
20. El-Sayed, A.M.A., Rida, S.Z., Arafa, A.A.M.: Exact solutions of the fractional-order biological population model. Commun. Theor. Phys. **52**, 992–996 (2009)
21. Rida, S.Z., El-Sayed, A.M.A., Arafa, A.A.M.: On the solutions of time-fractional 'reaction-diffusion equations. Commun. Nonlinear Sci. Numer. Simul. **15**, 3847–3854 (2010)
22. Rida, S.Z., El-Sayed, A.M.A., Arafa, A.A.M.: A fractional model for bacterial chemoattractant in a liquid medium. Nonlinear Sci. Lett. A **1**(4), 415–420 (2010)
23. Levine, H., Ben-Jacob, E.: Physical schemata underlying biological pattern formation-examples, issues and strategies. Phys. Biol. **1**, 14 (2004)
24. Adler, J.: Effect of amino acids and oxygen on chemotaxis on E. coli. J. Bacteriol. **92**, 121–129 (1966)
25. Adler, J.: Chemotaxis in bacteria. Science **153**, 225–234 (1971)
26. Golding, I., Kozlovsky, Y., Cohen, I., Ben-Jacob, E.: Studies of bacterial branching growth using reaction-diffusion models for colonial development. Physica A **260**, 510–554 (1998)
27. Adler, J.: Chemoreceptors in bacteria. Science **166**, 1588–1597 (1969)
28. Berg, H.C., Purcell, E.M.: Physics of chemoreception. Biophys. J. **20**, 193–219 (1977)
29. Lackie, J.M. (ed.): Biology of the Chemotactic Response. Cambridge Univ. Press, Cambridge (1986)
30. Blat, Y., Eisenbach, M.: Tar-dependent and -independent pattern formation by Salmonella typhimurium. J. Bacteriol. **177**, 1683–1691 (1995)
31. Miyata, S., Sasaki, T.: Asymptotic analysis of a chemotactic model of bacteria colonies. Math. Biol. **201**, 184–194 (2006)
32. Aslanidi, G.V., Aslanidi, O.V., Tsyganov, M.A., Holden, A.V., Ivanitsky, G.R.: Conditions causing wavefront instability in a growing colony of bacterial cells with chemotactic activity. Dokl. Biochem. Biophys. **394**, 18–20 (2004)
33. Keller, E.F., Segel, L.A.: J. Theor. Biol. **30**, 225–234 (1971)
34. Kawasaki, K., Mochizuki, A., Matsushita, M., Umeda, T., Shigesada, N.: Modeling spatio-temporal patterns created by Bacillus subtilis. J. Theor. Biol. **188**, 177–185 (1997)
35. Adomian, G.: Solving Frontier Problems of Physics: The Decomposition Method. Kluwer Academic, Dordrecht (1994)
36. Domain, G.A.: A review of the decomposition method in applied mathematics. Math. Anal. Appl. **135**, 501–544 (1988)
37. Munkhammar, J.D.: Fractional calculus and the Taylor–Riemann series. Undergrad J. Math. **6**, 1–19 (2005)
38. Cherruault, Y.: Convergence of Adomian's method. Kybernetes **18**, 31–38 (1989)
39. Cherruault, Y., Adomian, G.: Decomposition methods: a new proof of convergence. Math. Comput. Modell. **18**, 103–106 (1993)
40. Nagarhasta, N., Some, B., Abbaoui, K., Cherruault, Y.: New numerical study of Adomian method applied to a diffusion model. Kybernetes **31**, 61 (2002)
41. Momani, S.: Non-perturbative analytical solutions of the space- and time-fractional Burgers equations. Chaos Solitons Fractals **28**, 930–937 (2006)
42. Momani, S., Odibat, Z.: Analytical solution of a time-fractional Navier–Stokes equation by Adomian decomposition method. Appl. Math. Comput. **177**, 488–494 (2006)

43. Momani, S., Odibat, Z.: Numerical comparison of methods for solving linear differential equations of fractional order. Chaos Solitons Fractals **31**, 1248–1255 (2007)
44. Odibat, Z., Momani, S.: Approximate solutions for boundary value problems of time-fractional wave equation. Appl. Math. Comput. **181**, 767–774 (2006)
45. Podlubny, I.: Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. Academic Press, New York (1999)
46. Wazwaz, A.: A new algorithm for calculating Adomian polynomials for nonlinear operators. Appl. Math. Comput. **111**, 53–69 (2000)